Systematic renormalization in Hamiltonian light-front field theory: The massive generalization

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Hamiltonian light-front field theory can be used to solve for hadron states in QCD. To this end, a method has been developed for systematic renormalization of Hamiltonian light-front field theories, with the hope of applying the method to QCD. It assumed massless particles, so its immediate application to QCD is limited to gluon states or states where quark masses can be neglected. This paper builds on the previous work by including particle masses nonperturbatively, which is necessary for a full treatment of QCD. We show that several subtle new issues are encountered when including masses nonperturbatively. The method with masses is algebraically and conceptually more difficult; however, we focus on how the methods differ. We demonstrate the method using massive ϕ^3 theory in $5+1$ dimensions, which has important similarities to QCD. [S0556-2821(99)04716-5]

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I. INTRODUCTION

The use of a Hamiltonian light-front formalism may simplify the solution of quantum chromodynamics (QCD) by allowing us to make a convergent expansion of hadron states in free-particle Fock-space sectors. The Fock-space expansion will rapidly converge if the Hamiltonian satisfies certain conditions [1]. The Hamiltonian can then be used to solve for approximate hadron states.

Inspired by the work of Dyson [2], Wilson [3], Glazek and Wilson [4], and Wegner [5], significant work has been done to perturbatively derive light-front Hamiltonians that satisfy these conditions in the full Fock-space, neglecting zero modes [6–13]. The method developed by Allen and Perry [1] includes the scale dependence of the coupling and can be used to systematically renormalize light-front Hamiltonians, fixing all noncanonical operators, in principle, to all orders.

In this method the theory is regulated by placing on the Hamiltonian a smooth cutoff on change in free mass. The cutoff violates several physical principles, preventing renormalization exclusively through the redefinition of masses and canonical couplings. Renormalization must be completed by requiring the Hamiltonian to produce cutoff-independent physical quantities, and by requiring it to obey the physical principles of the theory that are not violated by the cutoff. These requirements completely fix the Hamiltonian so that it will give results consistent with all the physical principles of the theory, even those violated by the cutoff. The most powerful characteristic of this approach is that it systematically ''repairs'' the theory and retains only the fundamental parameters of the theory.

We generalize this method of renormalization to include particle masses and demonstrate it using massive ϕ^3 theory in $5+1$ dimensions. This theory is asymptotically free and its diagram structure is similar to QCD, which make it a good perturbative development ground. It is straightforward to extend the method for massless theories developed in Ref. [1] to calculate QCD quantities for which particle masses are unimportant, such as the low-lying glueball spectrum [14]. In this paper, we show how to incorporate particle masses nonperturbatively as a necessary step toward a treatment of full QCD.

II. REVIEW OF GENERAL FORMALISM

In this section we introduce some of the notation developed in Ref. [1] and outline the method. The formalism that is necessary for a detailed understanding of this method but that we do not repeat in this paper can be found in this earlier work. This includes the use of a unitary transformation to determine how the Hamiltonian changes with the cutoff [5], the use of physical principles to restrict the form of the Hamiltonian, and the details of how to compute matrix elements of the Hamiltonian.

We want to find the regulated invariant-mass operator, $\mathcal{M}^2(g_\Lambda,m,\Lambda)$, which is trivially related to the Hamiltonian. It can be split into a free part (which contains implicit mass dependence) and an interacting part:

$$\mathcal{M}^2(g_\Lambda,m,\Lambda)=\mathcal{M}^2_{\rm free}+\mathcal{M}^2_{\rm int}(g_\Lambda,m,\Lambda). \tag{1}$$

Since the method treats $\mathcal{M}^2_{\rm int}(g_\Lambda,m,\Lambda)$ perturbatively, we put the particle-mass term in $\mathcal{M}^2_{\rm free}$, to treat it nonperturbatively; however, $\mathcal{M}^2_{\rm int}(g_\Lambda,m,\Lambda)$ will still have mass dependence. The matrix elements of $\mathcal{M}^2(g_\Lambda,m,\Lambda)$ are written

$$\begin{aligned}\langle F|\mathcal{M}^2(g_\Lambda,m,\Lambda)|I\rangle &= \langle F|\mathcal{M}^2_{\rm free}|I\rangle+\langle F|\mathcal{M}^2_{\rm int}(g_\Lambda,m,\Lambda)|I\rangle\\ &= M_F^2\langle F|I\rangle + e^{-\Delta_{FI}^2/\Lambda^4}\\ &\quad\times\langle F|V(g_\Lambda,m,\Lambda)|I\rangle,\end{aligned} \tag{2}$$

where $|F\rangle$ and $|I\rangle$ are eigenstates of the free invariant-mass operator with eigenvalues M_F^2 and M_I^2, and Δ_{FI} is the difference of these eigenvalues. $V(g_\Lambda,m,\Lambda)$ is the interacting

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part of the invariant-mass operator with the Gaussian cutoff factor removed and is called the ''reduced interaction.'' With light masses Δ_{FI} will be small in a limited part of the phase space. This means there will be nonperturbative effects that must be dealt with through the diagonalization of the mass operator rather than its perturbative renormalization.

We expand $V(g_\Lambda, m, \Lambda)$ in powers of the running coupling, g_Λ:

$$V(g_\Lambda, m, \Lambda) = \sum_{r=1}^{\infty} g_\Lambda^r V^{(r)}(m, \Lambda), \tag{3}$$

where $V^{(1)}$ is the canonical interaction and the $V^{(r\geqslant 2)}(m,\Lambda)$'s are noncanonical interactions. These noncanonical operators can be thought of as counterterms in a traditional approach. Note that g_Λ implicitly depends on m. We use a unitary transformation to relate $\mathcal{M}^2(g_\Lambda, m, \Lambda)$ to $\mathcal{M}^2(g_{\Lambda'}, m, \Lambda')$, where $\Lambda' > \Lambda$. This yields the relation

$$V^{(r)}(m,\Lambda) - V^{(r)}(m,\Lambda') = \delta V^{(r)}(m,\Lambda,\Lambda') - \sum_{s=2}^{r-1} B_{r-s,s} V^{(r-s)}(m,\Lambda), \tag{4}$$

where $\delta V^{(r)}(m,\Lambda,\Lambda')$ is the $\mathcal{O}(g_{\Lambda'}^r)$ change in the reduced interaction and the $B_{r-s,s}$'s are functions of m, Λ, and Λ' that contain information on the scale dependence of the coupling. Since the scale dependence of the reduced interaction comes from g_Λ and the $V^{(r)}(m,\Lambda)$'s [See Eq. (3)], Eq. (4) simply states that if we subtract from $\delta V^{(r)}(m,\Lambda,\Lambda')$ the contribution due to the scale dependence of the coupling, then we are left with the contribution due to the scale dependence of the $V^{(r)}(m,\Lambda)$'s.

If there is a part of $V^{(r)}(m,\Lambda)$ that is independent of the cutoff, it will cancel on the left-hand-side of Eq. (4). For this reason, we split $V^{(r)}(m,\Lambda)$ into a part that depends on the cutoff, $V_{\rm CD}^{(r)}(m,\Lambda)$, and a part that is independent of the cutoff, $V_{\rm CI}^{(r)}(m)$:

$$V^{(r)}(m,\Lambda) = V_{\rm CD}^{(r)}(m,\Lambda) + V_{\rm CI}^{(r)}(m). \tag{5}$$

This division can be made with no ambiguity because we are assuming approximate transverse locality. Solving for both $V_{\rm CD}^{(r)}(m,\Lambda)$ and $V_{\rm CI}^{(r)}(m)$ is necessary to find the invariant-mass operator.

III. ADDITION OF PARTICLE MASSES

In our renormalized scalar theory m is the physical particle mass to all orders in perturbation theory. In a confining theory m would be considered the particle mass in the zero-coupling limit. Since the mass is being treated nonperturbatively, it must be included in the free part of $\mathcal{M}^2(g_\Lambda, m, \Lambda)$ in Eq. (1). This leads to an altered unitary transformation and fundamental changes in the renormalization procedure.

The changes in the procedure are discussed in the next subsections. The redefinition of the coupling (Sec. III A) is straightforward. In Secs. III B and III C, we present the expressions for the matrix elements of $V_{\rm CD}^{(r)}(m,\Lambda)$ and $V_{\rm CI}^{(r)}(m)$, respectively. We also qualitatively discuss the additional steps that are required to interpret and use them in a massive theory.

A. Coupling

The canonical definition of the coupling is

$$g = [64\pi^5 p_1^+ \delta^{(5)}(p_1 - p_2 - p_3)]^{-1} \langle \phi_2\phi_3 | \mathcal{M}_{\rm can}^2 | \phi_1 \rangle|_{p_2=p_3}. \tag{6}$$

In the massive theory, we choose

$$\begin{aligned} g_\Lambda &= [64\pi^5 p_1^+ \delta^{(5)}(p_1 - p_2 - p_3)]^{-1} \exp\left(9\frac{m^4}{\Lambda^4}\right) \\ &\quad \times \langle \phi_2\phi_3 | \mathcal{M}^2(g_\Lambda, m, \Lambda) | \phi_1 \rangle|_{p_2=p_3} \\ &= [64\pi^5 p_1^+ \delta^{(5)}(p_1 - p_2 - p_3)]^{-1} \\ &\quad \times \langle \phi_2\phi_3 | V(g_\Lambda, m, \Lambda) | \phi_1 \rangle|_{p_2=p_3}, \end{aligned} \tag{7}$$

which differs from the definition in the massless theory by the factor $\exp(9(m^4/\Lambda^4))$. This choice of coupling cancels a cutoff factor due to the presence of the mass and allows us to closely follow the formalism developed in the massless theory. In particular, the expressions for the matrix elements of $V_{\rm CD}^{(r)}(m,\Lambda)$ and $V_{\rm CI}^{(r)}(m)$ presented below have the same form as those derived in Ref. [1].

B. Cutoff-dependent contributions to $V^{(r)}(m,\Lambda)$

Momentum conservation implies that any matrix element of $V^{(r)}(m,\Lambda)$ contains a sum of terms, each with a unique product of momentum-conserving δ functions. Assuming that approximate transverse locality is maintained, the coefficient of each product of δ functions can be written as an expansion in powers of transverse momenta. In massive ϕ^3 theory, we can also make a generalized expansion in powers and logarithms of m. The scale dependence of any term in this expansion has the form

$$\Lambda^{6-2N_{\rm int}} \left(\frac{m}{\Lambda}\right)^\alpha \left[\ln\frac{m}{\Lambda}\right]^\beta \left(\frac{p_\perp}{\Lambda}\right)^\gamma, \tag{8}$$

where $N_{\rm int}$ is the total number of particles in the final and initial states that participate in the interaction. Also α, β, and γ are non-negative integers. For simplicity we display one component of transverse momentum, $p_\perp$; however, the general form includes a product of all transverse components from all particles. In principle, the introduction of a particle mass allows any function of m/Λ to appear. However, to $\mathcal{O}(g_\Lambda^3)$ the only extra scale dependence comes in the form $(m/\Lambda)^\alpha[\ln(m/\Lambda)]^\beta$. If $\beta=0$ and

$$6 - 2N_{\rm int} - \alpha - \gamma = 0, \tag{9}$$

the term is independent of the cutoff and is referred to as a ''cutoff-independent'' contribution. These contributions are discussed in the next subsection.

The expression for a matrix element of $V_{\rm CD}^{(r)}(m,\Lambda)$ is derived from Eq. (4):

$$\langle F|V_{\rm CD}^{(r)}(m,\Lambda)|I\rangle=\Bigg[\langle F|\delta V^{(r)}(m,\Lambda,\Lambda')|I\rangle - \sum_{s=2}^{r-1} B_{r-s,s}\langle F|V^{(r-s)}(m,\Lambda)|I\rangle\Bigg]_{\Lambda\ \rm terms}. \tag{10}$$

''Λ terms'' means the terms in the momentum and mass expansion that contain Λ' are to be removed from the expression in brackets. In terms that depend on positive powers of Λ', we do this by letting $\Lambda'\to 0$, and in terms that depend on negative powers of Λ', we let $\Lambda'\to\infty$.

C. Cutoff-independent contributions to $V^{(r)}(m,\Lambda)$

Considering the condition in Eq. (9), only two-point and three-point interactions can have cutoff-independent contributions. The lowest-order cutoff-independent three-point interaction is $V_{\rm CI}^{(3)}(m)$ and has not been explicitly computed in the massless or massive theories. However, $V_{\rm CI}^{(2)}(m)$ is the lowest-order cutoff-independent two-point interaction and must be calculated before anything is calculated to third order.

The matrix elements of $V_{\rm CI}^{(r)}(m)$ are divided into two-point and three-point contributions, and are given by the expression

$$\langle F|V_{\rm CI}^{(r)}(m)|I\rangle=\frac{1}{B_{r,2}}\Bigg[\langle F|\delta V^{(r+2)}(m,\Lambda,\Lambda')|I\rangle - \sum_{s=3}^{r+1} B_{r+2-s,s}\langle F|V^{(r+2-s)}(m)|I\rangle\Bigg]^{3\ \rm point}_{m^0\vec{p}_1^{\,0}\ \rm term} + \frac{1}{B_{r,2}}\Bigg[\langle F|\delta V^{(r+2)}(m,\Lambda,\Lambda')|I\rangle - \sum_{s=3}^{r+1} B_{r+2-s,s}\langle F|V^{(r+2-s)}(m)|I\rangle\Bigg]^{2\ \rm point}_{m^2\ \rm term}. \tag{11}$$

Here, ''$m^0\vec{p}_\perp^{\,0}$ term'' and ''m^2 term'' means expand the term in brackets in powers of external transverse momenta and in powers and logs of m, and keep only the term that is proportional to $m^0\vec{p}_\perp^{\,0}$ or m^2, respectively.[1] The removal of Λ and Λ' dependence is guaranteed by construction.

[1]The two-point contribution is independent of $\vec{p}_\perp$ because of cluster decomposition, transverse rotational invariance, and boost invariance.

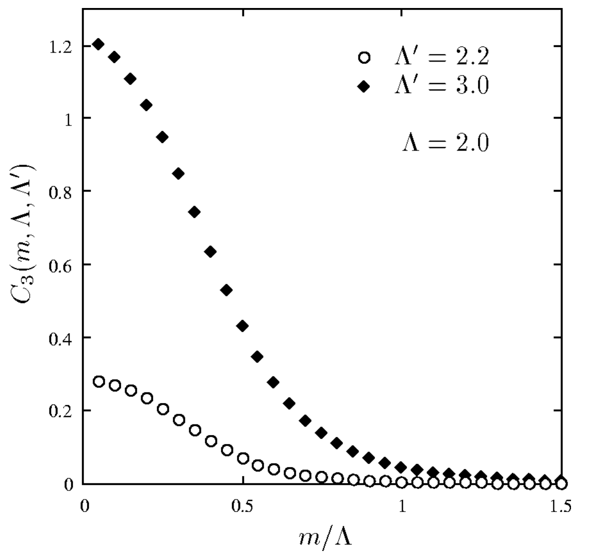

FIG. 1. The third-order coefficient of the running coupling as a function of the particle mass. Curves for various upper cutoffs with fixed lower cutoff show the coupling is exponentially damped with increasing mass.

Initially Eq. (11) looks useless because $V_{\rm CI}^{(r)}(m)$ depends on $V_{\rm CI}^{(r+1)}(m)$ [which is inside an integral in $\delta V^{(r+2)}$], suggesting the theory must be solved to all orders simultaneously. However, contributions to the reduced interaction from three-point interactions can only appear at odd orders, and contributions from two-point interactions can appear only at even orders. Thus, in the massless theory, this apparent problem does not manifest itself because there are no cutoff-independent two-point interactions. In the massive theory, although there are cutoff-independent two-point interactions, it is possible to solve for $V_{\rm CI}^{(2)}(m)$ and $V_{\rm CI}^{(3)}(m)$

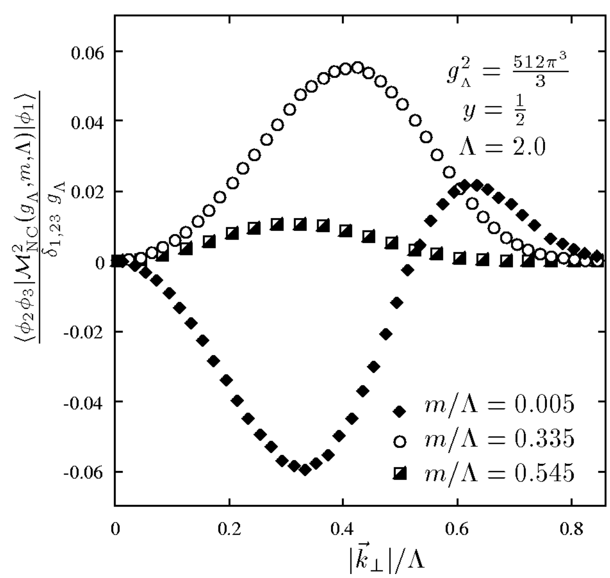

FIG. 2. The matrix element of the noncanonical part of the invariant-mass operator for $\phi_1\to\phi_2\phi_3$ versus the magnitude of the transverse momentum in the center-of-momentum frame. y is the longitudinal momentum fraction carried by particle 2.

simultaneously, without considering higher orders. This even-order/odd-order solution pattern can be extended to all orders.

Including self-energy contributions, the theory we want to describe contains particles of mass *m*. We can simplify the problem by using this fact instead of using Eq. (11) to solve for the even-order $V_{\rm CI}^{(r)}(m)$'s. We do this by forcing the completely disconnected parts of the forward *T*-matrix elements to be zero. (This part of a *T*-matrix element contains initial and final states that have the same number of particles *n* and *n* momentum-conserving δ functions.) This fixes the even-order $V_{\rm CI}^{(r)}(m)$'s since they only involve interactions on single particle lines. This allows us to calculate $V_{\rm CI}^{(2)}(m)$ independently of $V_{\rm CI}^{(3)}(m)$. This extra condition can be used to fix all even-order $V_{\rm CI}^{(r)}(m)$'s.

IV. RESULTS

The coupling in this theory runs at third order. We can compare the coupling at two different scales, Λ and Λ':

$$g_\Lambda = g_{\Lambda'} + \sum_{s=3}^{\infty} g_{\Lambda'}^s C_s(m,\Lambda,\Lambda'). \tag{12}$$

We can determine how the coupling runs at third order by solving for $C_3(m,\Lambda,\Lambda')$ (which is proportional to the matrix element $\langle \phi_2\phi_3|\delta V^{(3)}(m,\Lambda,\Lambda')|\phi_1\rangle|_{p_2=p_3}$). Figure 1 shows how $C_3(m,\Lambda,\Lambda')$ depends on the mass. The running of the coupling is exponentially damped as the mass grows since the cutoff inhibits production of intermediate particles. The difference between the values of the running coupling at two different scales increases as the two scales are separated. This is shown by the larger magnitude of $C_3(m,\Lambda,\Lambda')$ as the separation between Λ and Λ' grows.

Determining $V_{\rm CI}^{(3)}(m)$ requires a fifth-order calculation and is not attempted. However, calculating the matrix element $\langle \phi_2\phi_3|V_{\rm CD}^{(3)}(m,\Lambda)|\phi_1\rangle$ gives the relative sizes of the noncanonical interactions and the canonical interaction. Their relative magnitudes are similar to those in Ref. [1], suggesting that an expansion of the reduced interaction in powers of the running coupling is valid through third order.

Figure 2 shows how the noncanonical part of the matrix element of the invariant-mass operator for the interaction $\phi_1 \rightarrow \phi_2\phi_3$ depends on the magnitude of the transverse momentum in the center-of-momentum frame. Increasing the transverse momentum in the center-of-momentum frame increases the free mass of the system.

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